

## LETTERS TO THE EDITOR

## RIVER MEANDERS AND VORTICITY THEOREM

The serpentine windings of rivers, which in certain reaches attain a remarkable symmetry and are then called meanders, must represent one aspect of the tendency for establishment of quasi-equilibrium in natural streams. With the increase in knowledge of the interaction of forces operating in rivers, this tendency toward equilibrium has assumed a greater significance.

The current interest in these equilibrium processes is indicated by numerous recent papers dealing with the work of streams. The problem of river meanders has figured prominently in this recent literature. Discussions of meander mechanics have presented some conditions which are necessary but not sufficient to explain the origin of the serpentine curves of rivers.

A theory of meanders must provide an explanation for a number of observed relations. John P. Miller, in a personal communication, reports that while working on the glaciers of Alaska he observed streams of clear water, that is, without sediment load, which developed true meanders under some circumstances. Many tidal estuaries also meander where sediment load is negligible. The mechanics of meanders must, therefore, include some physical principle which does not depend on a load of debris. An adequate theory should explain why some rivers meander and others do not, and why meanders tend to form predominantly in downstream reaches rather than in the headwaters. It is logical also to expect that a given isolated curve in an otherwise straight channel imposes some effect on the flow. This effect must carry into the initially straight reach downstream a tendency for the development of a reverse curve; that is, a curve in the opposite direction from the initial one. A meander theory should provide such a mechanism.

The present note discusses in preliminary form a theory of the mechanics of river bends. The hypothesis is based on an application of the theorem of Kelvin that the absolute circulation of a closed fluid curve is conserved in an autobarotropic fluid. Though the hypothesis does not give definitive answers to the questions stated above, it at least provides some reasoning about each and suggests a direction for further work.

Conservation of angular momentum in a two-dimensional case--When water flows in a curved flume, the tendency for conservation of angular momentum can be observed providing that momentum is not dissipated through turbulent eddies or bed friction. A deep, narrow flume satisfies this condition. In the curved portion of such a flume, angular momentum is conserved and consequently the velocity will be distributed laterally across the flume cross-section in accordance with the equation

$$vr = \text{constant}$$

where  $v$  is the velocity at any point and  $r$  is the radius of curvature of the streamline at that point. This distribution of velocity is called "potential flow" and represents a condition in a fluid analogous to the conservation of angular momentum in mechanics.

This was demonstrated by WATTENDORF [1935] using a flume deep enough with respect to width for bed effects to be nearly eliminated. He showed that water flowing from a straight into a curved reach conserves its angular momentum and the velocity becomes redistributed so that the highest velocity is near the inside or convex wall where the radius of curvature of the streamline is shortest. There is thus a velocity gradient across the channel or a horizontal shear in the velocity field.

It is reasoned, therefore, that if the water leaving the curved reach of a meander enters the succeeding straight reach (crossover or point of inflection of the wave) with shear in the velocity field, this shear existing in the straight reach would, by the conservation of circulation, tend to cause the flow to develop curvature. The curve would be of such sense that the center of curvature would be on the side of the highest velocity; that is, the velocity gradient would be directed away from the center of curvature. It is necessary, then, to demonstrate that in the crossover reach of

a natural river meander, the circulation vector is not zero.

Conservation of circulation in fluid flow--The actual situation in a river meander is more complicated, however. Bed friction causes the surface streamlines to move across the stream away from the center of curvature while the streamlines near the bed turn toward the inner or convex bank.

The circulation is then not merely shear in the plan view of the velocity field, but consists of shear in each of three planes. Moreover, the cross-sectional area of flowing water changes from crossover to the center line of the bend and thus leads to divergence and convergence which affect the flow pattern.

The basic postulate made here is that in the flow in a meandering river, the circulation is conserved, or

$$dC/dt = 0$$

where C is circulation having the units of moment of momentum or ft<sup>2</sup>/sec. Let vorticity  $\omega$  be defined as

$$\omega = \delta C / \delta A$$

or circulation per unit area. Vorticity  $\omega$  can be resolved into three components,  $\xi$ ,  $\eta$ , and  $\zeta$  about the axes x, y, and z, where

$$\xi = \partial v_y / \partial z - \partial v_z / \partial y$$

$$\eta = \partial v_x / \partial z - \partial v_z / \partial x$$

$$\zeta = \partial v_y / \partial x - \partial v_x / \partial y$$

where  $v_x$ ,  $v_y$ , and  $v_z$  are the components of velocity in the x, y, and z directions.

It can be shown that if circulation is conserved or

$$dC/dt = 0$$

then the rate of change of vorticity equals the product of vorticity times divergence, with a minus sign,

$$d\omega/dt = -\omega [\partial(\rho v_x)/\partial x + \partial(\rho v_y)/\partial y + \partial(\rho v_z)/\partial z]$$

or

$$d\omega/dt = -\omega(\text{div } V)$$

It is further necessary, therefore, to prove that this equality is actually maintained at various points along the meander wave in a natural river.

Application to a natural meander--To test the hypothesis that the equation is satisfied, the author measured, with the help of John P. Miller, the horizontal velocity components in a natural meander of Baldwin Creek near Lander, Wyoming. The total horizontal velocity vector at a given point was measured by current meter, and its direction noted by a specially constructed direction vane attached to the wading rod. Baldwin Creek at the time of measurement carried 36 cfs, had a width of 20 ft, an average depth of one to two feet, and the meander wave length was 160 ft, amplitude 130 ft.

For the numerical integration the wave length was divided into 16 reaches along the stream. The width of the stream was divided into five parts, and horizontal slabs 0.4 ft in vertical thickness were used. Thus the flow was described by velocity through each of the six sides of unit volumes about 20 ft long, 4 ft wide, and 0.4 ft deep. Numerical integration was made with the assistance of M. Gordon Wolman.

The scalar values of the two sides of the equation

$$d\omega/dt = -\omega(\text{div } V)$$

were computed and are plotted in Figure 1a on the ordinate scale. The values of  $d\omega/dt$  are represented as open circles and of  $-\omega(\text{div } V)$  as crosses. The abscissa represents the positions along the meander wave. If the postulate is true that circulation is conserved in a natural meander, the crosses and circles should coincide on the ordinate scale. Though the correspondence is not exact, the values are of the same order and change sign along the meander wave in a remarkably consistent manner. It appears, then, that within the limit of error of the field data and integration method, circulation is conserved in the natural meander studied.

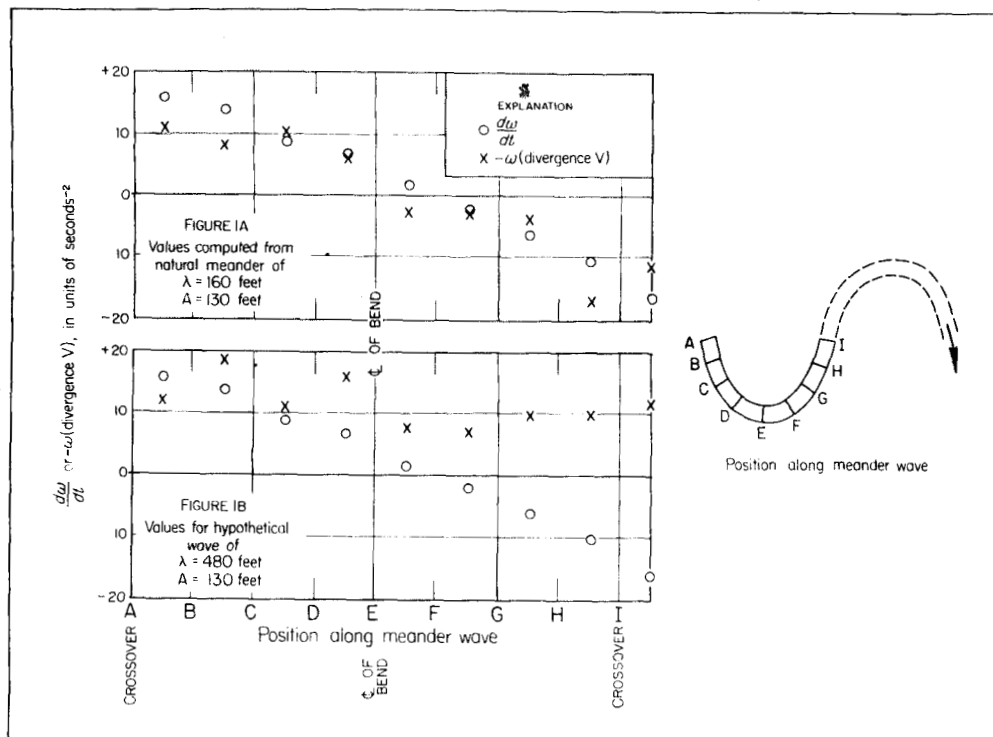


Fig. 1--Variation of the quantities  $d\omega/dt$  (open circles) and  $-\omega(\text{div } V)$  (crosses) along the meander wave; (a) data for a natural meander in which the two quantities are roughly equal; (b) data for a hypothetical wave having a larger wave length shows inequality of values

Using the same velocity distribution, the quantities were recomputed along a hypothetical meander wave having the same amplitude but a wave length of 480 ft, three times the actual wave length in the real river. The plot of  $d\omega/dt$  and of  $-\omega(\text{div } V)$  versus position along the wave are shown in Figure 1b. It can be seen that the equality of values is not maintained; that is, circulation is not conserved. This suggests, then, that a natural meandering river forms meander waves of such a length that the circulation is conserved, and, therefore, the conservation of circulation is basic to meander formation.

The river data studied showed that velocity is not uniformly distributed in the cross section of the crossover, but rather the largest velocity is nearer the bank which was the outside or concave bank in the curve immediately upstream. If circulation is conserved the straight reach should be succeeded by a curve such that the center of curvature lies on the same side of the stream as the greatest velocity in the straight reach, just as in the Wattendorf flume. The present theory, then, provides a mechanism for the development of a curve below the point of inflection or crossover, and the curvature will be opposite in sign to that of the bend immediately upstream.

It is postulated that formation of meanders depends on the preservation of the circulation through

a reach of channel. This circulation would be dissipated by small scale eddies if the velocity-depth ratio were large, whereas it would be conserved rather than dissipated in a channel having a small value of the velocity-depth ratio. It has been demonstrated [LEOPOLD and MADDOCK, 1953] that in upstream reaches of rivers the velocity-depth ratio is large and decreases downstream. It is reasoned, therefore, that meanders occur primarily in the downstream reaches of rivers for that reason. This is in agreement with the observation that meanders occur in headwater tributaries only where depth is large relative to velocity.

A more complete treatment of this problem will be published after the investigation has been pursued further.

#### References

- LEOPOLD, L. B., and T. MADDOCK, The hydraulic geometry of stream channels and some physiographic implications, U. S. Geological Survey, Prof. Paper 252, 1953.  
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