

## GEOMETRY OF RIVER CHANNELS<sup>a</sup>

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Discussion by William W. Emmett and Luna B. Leopold

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WILLIAM W. EMMETT<sup>13</sup> AND LUNA B. LEOPOLD.<sup>14</sup>—For many years river engineers and geomorphologists have sought a rationale for the general similarity that can be observed among river channels from various environments. Some aspects of this general comparability were noted a century ago by Playfair, and other aspects were examined in a more quantitative way approximately 10 yr ago by R. E. Horton. Such similarities with respect to the channel cross section and the associated hydraulic parameters and their changes downstream were studied by some of the engineers who attained prominence for their work on the self-adjusting irrigation canals of India, especially G. Lacey,<sup>15</sup> C. Inglis,<sup>16</sup> and T. Blench.<sup>17</sup> Among these men, canal characteristics were the principal objects of investigation, although natural river channels assumed an increasing share of the attention of the students of "regime analysis," especially Blench. L. B. Leopold and T. Maddock<sup>18</sup> developed a scheme by which the average relationships among form and hydraulic parameters could be easily described, this description being the one on which the author chose to elaborate.

It would probably not be unfair to state that the students mentioned herein, and a few others, are far outnumbered by those who have devoted their energy to individual rivers or reaches in a given river, to specific floods, and to debris transport under laboratory conditions; in short, to studies that have focused on differences among rivers, rather than on similarities, or on the understanding of specific phenomena within sections of a river but not to the river as an entity. In fact, to many practical engineers faced with problems of regulation or development, the differences are far more important than the similarities.

Of course, the geologist, especially one concerned with geomorphology, has seen the scope of fluvial processes in landscape sculpture and could therefore conceive of the river as an entity. This is a view that demands attention to the general, rather than to the particular. But the geologist, at

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<sup>a</sup> March, 1964, by Walter B. Langbein (Proc. Paper 3846).

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<sup>15</sup> Lacey, G., "Stable Channels in Alluvium," *Proceedings, Institution of Civ. Engrs.*, London, England, Vol. 229, Part 1, 1930, pp. 259-384.

<sup>16</sup> Inglis, C., "The Behavior and Control of Rivers and Canals," *Publication 13, Central Water Power Irrigation and Navigation Research Sta., Poona, India, Vol. 2, 1949.*

<sup>17</sup> Blench, T., "Hydraulics of Sediment-Bearing Canals and Rivers," *Evans Industries, Ltd., Vancouver, B. C., Canada, 1951.*

<sup>18</sup> Leopold, L. B., and Maddock, T., "The Hydraulic Geometry of River Channels and Some Physiographic Implications," *Professional Paper 252, Geol. Survey, U. S. Dept. of the Interior, Washington, D. C., 1954.*

least until recently, has either been content to deal with the matter in qualitative terms or has been too little acquainted with the hydraulic aspects of the problem to do otherwise. This situation is rapidly improving. In fact, the modern geologist is becoming so specialized and so well-trained in basic sciences that a new danger presents itself; that is, the possibility that he might abandon his heritage as a natural scientist in the broad sense.

There has been, then, a notable difference in viewpoint in studies of river characteristics. One viewpoint overlooks the rather appreciable differences from one natural channel to another in order to concentrate on the generalities; the other viewpoint, although aware of the existence of similarities among rivers, has chosen to examine the problem through the study of individual cases or of particular processes within a channel.

The existence of different approaches carries an implication of considerable moment. The writers conceive this to be as follows: Natural river channels are characterized by similarities and by differences, and both are important to the understanding of the river system. Any generalization that purports to explain the ubiquitous features of river channels must also allow, indeed incorporate, a correlative provision for the formidable differences observed among reaches of the same river and among rivers in a common environment.

It is in this respect that the theory presented by the author appears to be more advanced than any previous analysis, while, at the same time, he derives quantitative values of parameters that have been recognized by many previous studies as descriptive of important general features of rivers. Moreover, Langbein has shown the same type of theoretical analysis to be capable of yielding quantitative agreement with certain results obtained in recirculating flumes.

In preceding paragraphs attention was called to certain dispersion of effort resulting from the fact that many students capable of seeing the broad problem of fluvial forms in landscape have, by interest and training, neglected the quantitative and the particular, whereas specialists and practical men, perhaps, have not been sufficiently aware of the need for generalization. But slow progress in that understanding needed in a spectrum of related sciences and in engineering may also stem from another circumstance. Authors may fail to negotiate successfully between the Scylla of brevity and elegance, which can suck a paper into incomprehensibility, and the Charybdis of excessive length, which may prevent publication in a widely read journal, or whose talons will forever keep the ponderous tome tightly shut on the library shelf. The writers believe that the paper lies perilously close to the former, partly because many potentially interested engineers will find the statistical argument difficult to follow, and partly because the paper contains no field examples and only the briefest reference to pertinent findings in the published literature. The writers recognize that many readers will at least be acquainted with these published data, and that a theoretical paper should not concern itself with the details of proof. But the writers feel that the author's theory warrants a wide audience, and, to this end, a numerical solution to his equations may make his reasoning more generally understood.

Langbein provides three physical relationships that must be fulfilled in the stream system; namely, continuity, a relation of friction to hydraulic factors, and a relation of sediment transport to hydraulic factors.

Continuity is expressed by

$$Q = w d v \dots\dots\dots (24)$$

An applicable friction formula is the Manning equation

$$v = \frac{1.49 d^{2/3} S^{1/2}}{n} \dots\dots\dots (25)$$

A relation of sediment load in terms of concentration of sediment in the sediment-water mixture, C, is derived from Bagnold<sup>19</sup>

$$C \propto \frac{(v d)^{1/2} S^{3/4}}{n^4} \dots\dots\dots (26)$$

Eqs. 24 to 26 can be written in a form that involves the exponents in the hydraulic geometry; that is,  $v \propto Q^m$ ,  $d \propto Q^f$ ,  $w \propto Q^b$ ,  $S \propto Q^z$ , and  $m \propto Q^y$ . Eqs. 24, 25, and 26 then become

$$b + m + f = 1 \dots\dots\dots (27)$$

$$m = \frac{2}{3}f + \frac{1}{2}z - y \dots\dots\dots (28)$$

and

$$\frac{1}{2}m + \frac{1}{2}f + \frac{3}{2}z - 4y = 0 \dots\dots\dots (29)$$

Combining these to eliminate z and solving for y yields

$$y = 3.5 m - 1.5 f \dots\dots\dots (30)$$

In addition to these three physical equations (Eqs. 27, 28, and 29) two additional postulates: (1) Rate of work per unit bed area is uniform as possible; and (2) rate of work in the whole system is also as small as possible. These are tendencies, in that they cannot be completely satisfied simultaneously. To the extent that rate of work (power expenditure) is equalized over all parts of the bed through the river system, total rate of work in the whole system is increased. Thus, one of these conditions is fulfilled only at the expense of the other. This indicates a logic behind the author's contention that the two will jointly be approached most equably when the variances of their respective values are minimized.

Considering the first postulate mentioned, power or rate of doing work in a unit length of stream channel is the product of discharge and slope, Q s, per

<sup>19</sup> Bagnold, R. A., "Sediment Discharge and Stream Power—A Preliminary Announcement," Circular 421, Geol. Survey, U. S. Dept. of the Interior, Washington, D. C., 1960.

unit width (thus per unit area of channel), the power is  $Q s/w$ . So that this term is as uniform as possible, it is written as

$$\frac{w \, d \, v}{w} \, s = \frac{Q \, s}{w} \rightarrow \text{constant}, \dots \dots \dots (31)$$

in which the arrow indicates "tends to approach." In terms of exponents in the hydraulic geometry, Eq. 31 becomes

$$b + f + m + z - b = f + m + z - 0 \dots \dots \dots (32)$$

With regard to the second postulate, the tendency for minimum total work in the whole system would be completely fulfilled if, in each unit length along the channel, the increase in discharge resulting from tributary entrance were accompanied by a proportional decrease in slope, so that for each reach or

TABLE 1.—HYPOTHETICAL EXAMPLES

Reach number (1)	Discharge, Q, in reach (2)	Case 1 z = 0		Case 2 z = -0.5		Case 3 z = -0.75		Case 4 z = -1.0 <sup>a</sup>	
		s	Q s	s	Q s	s	Q s	s	Q s
		(3)		(4)		(5)		(6)	
1	1	2	2	3.5	3.5	4.2	4.2	4.96	5.0
2	2	2	4	2.4	4.8	2.5	5.0	2.48	5.0
3	4	2	8	1.7	6.8	1.45	5.8	1.24	5.0
4	6	2	12	1.4	8.4	1.1	6.6	.82	5.0
5	10	2	20	1.0	10	.75	7.5	.50	5.0
Totals		10	46	10.0	33.5	10.0	29.1	10.0	25.0

<sup>a</sup> The product  $Q s$  remains finite only for values of  $z > -1.0$ .

each unit of stream length, the product,  $Q s$ , has the same value. The power expended in the whole system is the summation of rates of work in each part, or total power

$$P = \Sigma Q_1 s_1 + Q_2 s_2 \dots Q_1 s_1 \dots \dots \dots (33)$$

in which the subscripts refer to different individual units of stream length. If it is not apparent that P is minimum when the products of  $\Delta Q \Delta s$  are equal, a simple numerical example may be useful. In Table 1, which shows how the downstream change of gradient, s, affects the power expended in various reaches and the total power expenditure,  $Q s$  in the river system, the assumed basin relief is 10 units, and the discharge is assumed to go from 0 at the headwater to 10 units at the mouth, as shown in Column 2. The five reaches or unit distances downstream are arbitrarily assigned values of s. The value s in a reach is the drop in elevation in the reach.

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The sum of the elevation drops,  $s$ , in each case is 10, as required by the assumptions. The products,  $Q s$ , are totaled for each case, and it can be seen that when  $s$  is chosen to decrease in proportion to the increase in discharge (Case 4), the products,  $Q s$ , are equal in each reach, and the sum  $\Sigma Q s$  is minimum.

Thus, minimum power is expended in the system when  $Q s$  is equal in each reach or

$$Q s = \text{constant} \dots\dots\dots (34)$$

In terms of exponents,

$$b + f + m + z = 0 \dots\dots\dots (35)$$

but

$$b + f + m = 1 \dots\dots\dots (36)$$

so

$$1 + z = 0 \dots\dots\dots (37)$$

In addition, Eq. 30 must be satisfied. There are two conditions that cannot be met completely, but the most probable solution is sought between them. The author has shown how this most probable condition can be ascertained.

In the hope of further clarification, subsequent computations show how the most probable values can be obtained graphically. The latter can be accomplished by computing values of the variances of the terms  $f + m + z$  and  $1 + z$ . If the variances of such terms are proportional to the squares of the terms themselves, then the most probable values will be obtained by minimizing the square root of the sums of squares of the terms, or

$$\sqrt{(f + m + z)^2 + (1 + z)^2} = \text{minimum} \dots\dots\dots (38)$$

Values of this quantity can be computed and plotted, and the minimum condition can be noted. By assuming values of  $f$  and  $m$ , corresponding values of  $z$  can be obtained from  $m = 2/3 f + 1/2 z - y$ , and values of  $y$  can be obtained from Eq. 30. Also,  $b = 1 - f - m$ .

The values of the factors that provide the minimum variance (minimum value of the quantity under the radical) represent the condition that is the most probable balance between tendencies for minimum power expenditure per unit bed area and minimum work rate in the whole river system. But, in addition, as the author states, "the relative changes in velocity, depth, and width that accompany a given range of power . . . are most nearly equal when the sums of squares of the hydraulic indices is a minimum." That is, in the downstream river case considered herein, all hydraulic indices participate equally in adjustment to accommodate the increase of discharge downstream. No single hydraulic parameter takes up all, or an undue share, of the downstream change required. This specification is defined by minimizing the

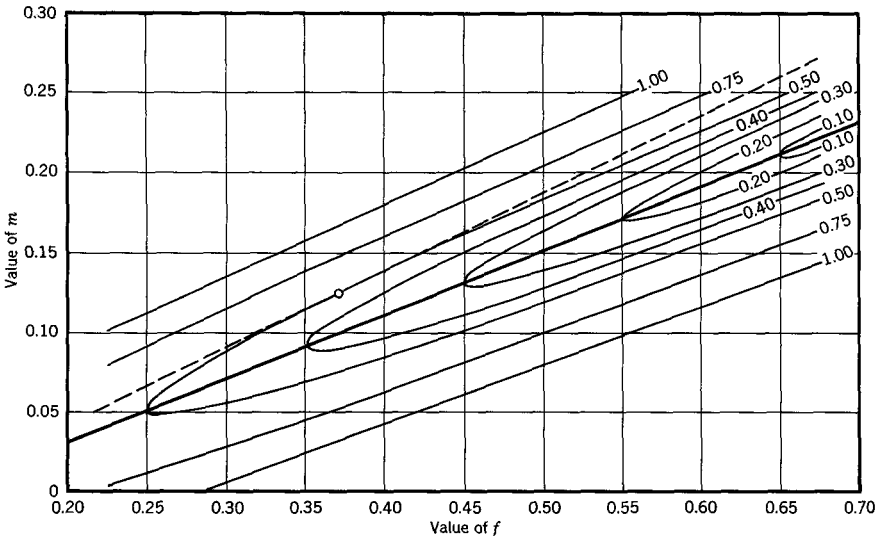


FIG. 1.—VALUES OF  $\sqrt{(m + f + z)^2 + (1 + z)^2}$  AS A FUNCTION OF m AND f

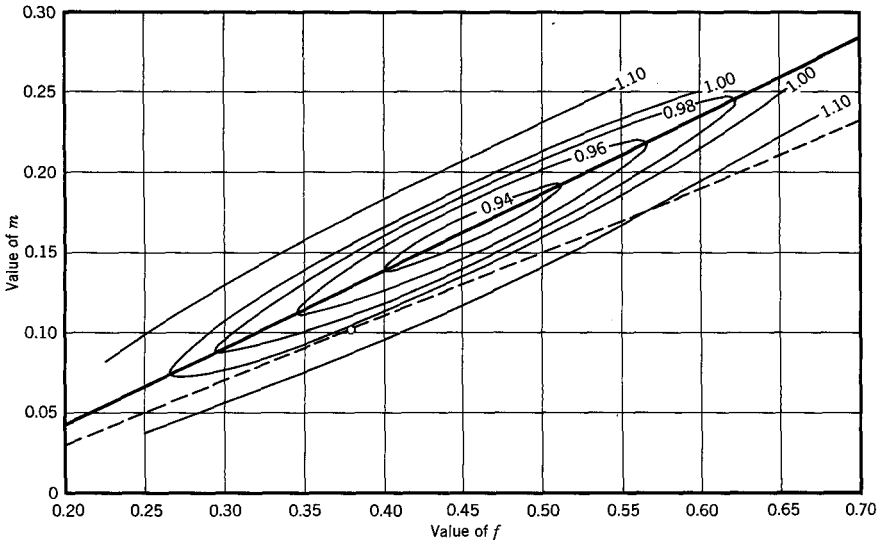


FIG. 2.—VALUES OF  $\sqrt{m^2 + f^2 + b^2 + z^2 + (1 + z)^2}$  AS A FUNCTION OF m AND f

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$$\sqrt{m^2 + f^2 + b^2 + z^2 + (1 + z)^2} = \text{minimum} \dots \dots \dots (39)$$

For ease of computations, a table was set up, in which various combinations of values of  $m$  and  $f$  were chosen, and from the equations mentioned earlier, values were computed for  $b$ ,  $y$ ,  $z$ ,  $\sqrt{(f + m + z)^2 + (1 + z)^2}$ , and  $\sqrt{m^2 + f^2 + b^2 + z^2 + (1 + z)^2}$ .

Fig. 1 is a plot of values of  $\sqrt{(f + m + z)^2 + (1 + z)^2}$  as a function of  $m$  and  $f$ . It can be seen that minimum values of this variance parameter form a

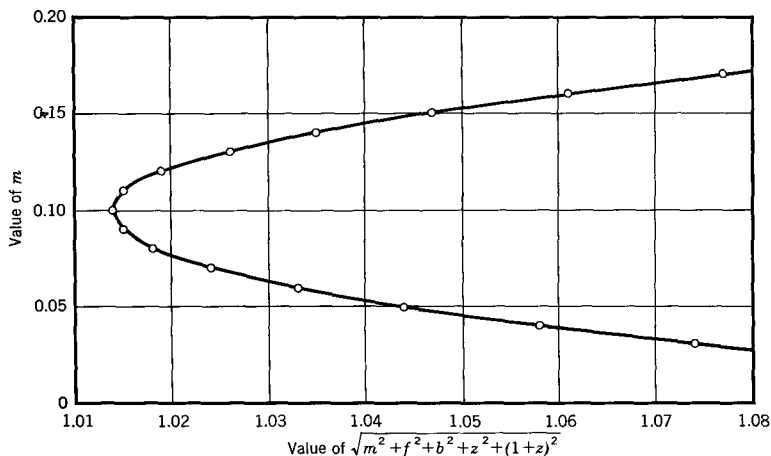


FIG. 3.—SOLUTION FOR CENTRAL VALUE OF  $m$

locus of points defined by the heavy line on Fig. 1., the equation of which is determined from the graph to be

$$m = -0.05 + 0.40 f \dots \dots \dots (40)$$

But the desired pair of values of  $m$  and  $f$  are not determined solely by minimization of this variance parameter; in addition, it is necessary to minimize, insofar as possible, the variance parameter  $\sqrt{m^2 + f^2 + b^2 + z^2 + (1 + z)^2}$ , values of which are shown plotted in terms of  $f$  and  $m$  on Fig. 2. Its lowest values also form a locus of points that is shown as a heavy line on Fig. 2. On Fig. 2 is also copied the locus of minima defined by Fig. 1, Eq. 40, shown as a dashed line.

The most probable desired values of  $m$  and  $f$  may be read from Fig. 2 as the position that lies on the locus of points defined by the dashed line and for which the value of  $\sqrt{m^2 + f^2 + b^2 + z^2 + (1 + z)^2}$  is minimum. This is the

point at which as low a value of the elliptical isopleths of Fig. 2 is tangent to the dashed line, that point being marked by the small circle at  $f = 0.38$ ,  $m = 0.10$ . It will be seen, however, that the point of tangency is insensitive.

By the same reasoning, the values of  $m$  and  $f$  should be shown on Fig. 1 at a point on the locus of minima from Fig. 2, at which isopleths of  $\sqrt{(m + f + z)^2 + (1 + z)^2}$  are minimized. To that end, the heavy line of Fig. 2 is plotted as a dashed line on Fig. 1. The required point of tangency is again insensitively defined, but appears to be at approximately the position of  $f = 0.37$ ,  $m = 0.125$ , approximately the same as obtained from the other graph.

The most exact solution for determining the minimum value of  $\sqrt{m^2 + f^2 + b^2 + z^2 + (1 + z)^2}$ , with the consideration that values of  $m$  and  $f$  must fit Eq. 40, is to plot values of the former against values of  $m$  determined by Eq. 40. These values are plotted in Fig. 3. The abscissa scale has been greatly exaggerated to define the minimum value more accurately. The plotted data indicate minimization at a value of  $m = 0.10$ . Values of the other exponents may be computed using the outline presented in the table, and these are

$$\left. \begin{aligned}
 m &= 0.10 \\
 f &= 0.37 \\
 b &= 0.53 \\
 z &= -0.72 \\
 y &= -0.21
 \end{aligned} \right\} \dots\dots\dots (41)$$

The respective values obtained by the author, who used an analytic, rather than a graphical solution, were 0.10, 0.37, 0.53, and -0.73. In the case of the alternate solution, in which  $f = 0.37$  and  $m = .125$ , the values of the remaining factors are

$$\left. \begin{aligned}
 b &= 0.50 \\
 z &= -0.50 \\
 y &= -0.12
 \end{aligned} \right\} \dots\dots\dots (42)$$

It will further be recalled that the field data from rivers published by Leopold and Maddock<sup>18</sup> were 0.10, 0.40, 0.50 for  $m$ ,  $f$ , and  $b$ . For  $z$ , the published figures from field data have ranged from -0.49 to -0.95.

Fig. 3 also illustrates the insensitiveness of the solution for values of  $m$  in the range of 0.07 to 0.13. However, this is partly obscured by the exaggerated abscissa scale. This again indicates that river channels may have considerable leeway in adjusting to changes in stream power and still follow the behavior predicted by the author's theory remarkably well.

The author has presented a rationale for river channel adjustment that allows quantitative computation of important hydraulic and form parameters that, insofar as they have been adequately observed in the field data, are in good agreement with field data. Furthermore, the theory shows that there should be an insensitivity of the adjustment toward equilibrium, a characteristic that explains the general similarity, as well as the individual variability



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among rivers. The writers believe that Langbein's contribution will be the most important advance of the present (1964) generation in the understanding of river morphology.