Convective heat transfer in planetary dynamo models

Eric M. King and Krista M. Soderlund
Department of Earth and Space Sciences, University of California, Los Angeles, California 90095-1567, USA
(eric.king@ucla.edu)

Ulrich R. Christensen and Johannes Wicht
Max-Planck-Institut für Sonnensystemforschung, D-37191 Katlenburg-Lindau, Germany

Jonathan M. Aurnou
Department of Earth and Space Sciences, University of California, Los Angeles, California 90095-1567, USA

[1] The magnetic fields of planets and stars are generated by the motions of electrically conducting fluids within them. These fluid motions are thought to be driven by convective processes, as internal heat is transported outward. The efficiency with which heat is transferred by convection is integral in understanding dynamo processes. Several heat transfer scaling laws have been proposed, but the range of parameter space to which they apply has not been firmly established. Following the plane layer convection study by King et al. (2009), we explore a broad range of buoyancy forcing \( Ra \) and rotation strength \( E^{-1} \) to show that heat transfer \( Nu \) in spherical dynamo simulations occurs in two distinct regimes. We argue that heat transfer scales as \( Nu \sim Ra^{6/5} \) in the rapidly rotating regime and \( Nu \sim Ra^{2/7} \) in the weakly rotating regime. The transition between these two regimes is controlled by the competition between the thermal and viscous boundary layers. Boundary layer scaling theory allows us to predict that the transition between the regimes occurs at a transitional Rayleigh number, \( Ra_t = E^{-7/4} \). Furthermore, boundary layer control of heat transfer is shown to relate to the interior temperature profiles of the models. In the weakly rotating regime, the interior fluid is nearly adiabatic. In the rapidly rotating regime, adverse mean temperature gradients abide, irrespective of the Reynolds number \( Re \). Extrapolating our results to Earth’s core, we estimate that core convection resides in the rapidly rotating regime, with \( Ra \approx 2 \times 10^{24} \) \( (Ra/Ra_t \approx 0.02) \), corresponding to a superadiabatic density variation of \( \Delta \rho/\rho_o \approx 10^{-7} \), which is significantly below the sensitivity of present seismic observations.

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1. Introduction

[2] Many astrophysical and geophysical bodies contain large quantities of electrically conducting fluids: plasma in the Sun; liquid metal in the cores of Earth and other terrestrial worlds; metalized hydrogen gas in Jupiter and Saturn; and ionic fluids in Uranus and Neptune [Gubbins and Roberts, 1987; Stevenson, 2003; Miesch, 2005]. Motions within these fluids generate electrical current systems that power the magnetic fields we observe on these bodies. The conversion of kinetic energy into electromagnetic energy is known as dynamo action. The kinetic energy needed to fuel dynamo action is widely thought to be driven by convection in Earth’s core, within the Sun, as well as within the Giant Planets. Most planetary dynamo models then consist of a spherical shell of conducting fluid, with thermal energy deposited at the inner boundary and extracted from the outer. Unstable to this configuration, the fluid convects to transport heat outward and generates electrical currents that can produce self-sustaining magnetic fields.

[3] Geophysical and astrophysical bodies rotate, and this background rotation acts upon the convecting fluid via the Coriolis force. Rapidly rotating convection tends to produce well-organized, axially aligned flow structures. This organizational effect of the Coriolis force constrains the efficiency of heat transfer [e.g., Rossby, 1969; Liu and Ecke, 2009] and is thought to permit the generation of large-scale magnetic fields [Olson and Christensen, 2006; Käpylä et al., 2009]. When the system is forced with sufficient strength, or when the system rotates sufficiently slowly, convection can break free of the constraints of rotation, and results in more chaotic three dimensional turbulence [Gilman, 1977; Aurnou et al., 2007; King et al., 2009]. Such flows have been found to generate smaller-scale magnetic fields [Kutzner and Christensen, 2002; Olson and Christensen, 2006; Käpylä et al., 2009].

[4] In this paper, we explore the limits of rotational control of convection dynamics in planetary dynamo models via measurements of heat transfer efficiency in 200 numerical dynamos, spanning a broad range of parameter space. In section 2 we briefly review previous work on heat transfer in rotating convection. In section 3 we detail our numerical methodology. In section 4 we present heat transfer results from the dynamo models. In section 5 we discuss interior temperature profiles, and we apply these results to Earth’s core in section 6. Last, a summary is given in section 7.

2. Previous Studies of Rotating Convective Heat Transfer

2.1. Role of Boundary Layers

[5] The nondimensional parameters used in this study are defined in Table 1. The five base input parameters consist of buoyancy forcing, strength of rotation, fluid properties, and shell geometry. The strength of buoyancy forcing versus diffusive effects is characterized by the Rayleigh number, \( Ra \). The strength of the Coriolis force versus viscosity is characterized by the inverse of the Ekman number, \( E^{-1} \). The relative values of the fluid’s diffusivities are characterized by the thermal and magnetic Prandtl numbers, \( Pr \) and \( Pm \), respectively. The spherical shell geometry is characterized by the ratio between inner and outer radii, \( \chi \). The key output parameter used in this study is the Nusselt number, \( Nu \). The Nusselt number characterizes the efficiency of convective heat transfer as the ratio of total (superadiabatic) heat transfer to that transferred by diffusion alone.

[6] Our goal is to describe heat transfer behavior by assessing how \( Nu \) depends on the control parameters \( Ra, E, Pr, \) and \( Pm \). Studies of nonrotating (\( E = \infty \)), turbulent convection often seek to fit empirical data with power law scalings of heat transfer efficiency versus thermal driving,

\[ Nu \propto Ra^\alpha, \]

for a given fluid (fixed \( Pr, Pm \)). The influence of rotation adds a new degree of freedom to the system, spreading heat transfer data in two dimensions, \( Ra \) and \( E \). The goal of heat transfer scaling analysis is to collapse these heat transfer data. This is typically done by invoking some physically motivated combination of control parameters [e.g., Boubnov and Golitsyn, 1990; Canuto and Dubovikov, 1998; Schmitz and Tilgner, 2009; Liu and Ecke, 2009; King et al., 2009]. The comparison between theoretically founded scaling arguments and empirical data therefore allows us to test our understanding of convection dynamics. Furthermore, these scalings permit extrapolation of model behavior to the extreme parameter values of planets and stars, at which models cannot be directly evaluated.

[7] King et al. [2009] examine heat transfer behavior in plane layer, nonmagnetic, rotating convection and
find two distinct regimes (see Figure 1). In the weakly rotating regime, heat transfer is independent of the rotation rate, and scales as

$$Nu \sim Ra^{2/7},$$

similar to power law fits from the wealth of non-rotating convection studies [e.g., Castaing et al., 1989; Glazier et al., 1999]. This scaling is signified by the solid line in Figure 1. In the rapidly rotating regime, heat transfer data exhibits a steeper scaling behavior. The transition between these two heat transfer regimes is argued to be controlled by the relative thicknesses of the boundary layers. Boundary layers are thin regions of fluid immediately adjacent to the bounding surface where diffusive effects become important. Two boundary layers exist in the rotating convection system: the thermal boundary layer; and the viscous boundary layer, which is the so-called Ekman layer in Coriolis force-dominated systems. King et al. [2009]

Figure 1. Convective heat transfer data, $Nu$, versus $Ra$ in a rotating right cylinder of water with $E \approx 4 \times 10^{-5}$ (adapted from King et al. [2009]). The dotted vertical line indicates the transitional Rayleigh number, $Ra_c = Ra_i = E^{-7/4}$. The dashed line represents the rapidly rotating scaling, $Nu = (Ra/Ra_c)^{6/5}$, and the solid line indicates the nonrotating scaling, $Nu \sim Ra^{2/7}$. The rapidly rotating scaling is dependent on $Ra_c \sim E^{-4/3}$, which is the critical Rayleigh number predicting the onset of hydrodynamic convective instability [e.g., Jones et al., 2000].
suggest that when the Ekman layer is thinner than the thermal boundary layer, convective heat transfer is inhibited by the influence of rotation. Conversely, when the thermal boundary layer is thinner, the constraining influence of rotation on convection is lost, and so heat transfer conforms to the nonrotating behavior.

[3] Theoretical scalings of the boundary layer thicknesses permit a prediction of the transition point between the two regimes. The Ekman and thermal boundary layers have thicknesses

\[ \delta_E \sim E^{1/2} \quad \text{and} \quad \delta_\kappa \sim \xi^{-1}. \]  

respectively [Greenspan, 1968; Spiegel, 1971]. The transition is predicted to occur where the boundary layers cross, i.e., when \( \delta_E = \delta_\kappa \). The Nusselt number at which the transition should occur is then

\[ Nu \sim E^{-1/2}. \]  

The Nusselt number can be related to the Rayleigh number using the heat transfer scaling law \( Nu \sim Ra^{2/7} \). This allows us to solve for the Rayleigh number at which the boundary layers are predicted to cross:

\[ Ra_t = E^{-7/4}, \]  

which we call the transitional Rayleigh number. Note that in the work by King et al. [2009], there is a scaling prefactor on the right-hand side of (5) of 1.4. Here we set this prefactor to unity.

[9] We predict a rapidly rotating heat transfer scaling law using the boundary layer controlled transition point. First, we assume a rapidly rotating scaling law of the form \( Nu = (Ra/Ra_c)^7 \), where \( Ra_c \) is the critical Rayleigh number for the onset of convection, \( Ra_c \propto E^{-4/3} \) [Chandrasekhar, 1953]. In order for this scaling law to intersect with the weakly rotating power law (2) at the transitional Rayleigh number (5), the rapidly rotating scaling exponent must be \( \gamma = 6/5 \), yielding

\[ Nu = (Ra/Ra_c)^{6/5}. \]  

in the rotationally controlled convection regime. This scaling is signified by the dashed line in Figure 1. The introduction of the Coriolis force to this scaling through the critical Rayleigh number signifies the importance of rotation in this regime.

[10] The separation of rotating convection dynamics into rapidly rotating and weakly rotating regimes is not new [e.g., Gilman, 1977; Kutzner and Christensen, 2002; Aurnou et al., 2007; Driscoll and Olson, 2009]. Canuto and Dubovikov [1998], for example, work to distinguish two such regimes based on convective mixing length theory. Christensen and Aubert [2006] and Olson and Christensen [2006] use an a posteriori measurement of a local Rossby number to differentiate the two regimes in terms of magnetic field morphology in dynamo models. Schmitz and Tilgner [2009] separate the two regimes by comparing heat transfer with measurements of a modified Peclet number in plane layer rotating convection simulations. Here we test whether the boundary layer controlled transition predicted by King et al. [2009] adequately describes the transition between these two regimes in planetary dynamo models.

### 2.2. Relevance to Dynamo Models

[11] We hypothesize that boundary layer control of rotating convective heat transfer regimes applies to planetary dynamo models. There are several reasons why the plane layer results may fail to apply to planetary and stellar dynamos. For instance, the added complexity of a spherical geometry and the additional dynamic influence of the magnetic field via Lorentz forces were not present in the plane layer study.

[12] In spherical geometries, Ekman boundary layers have latitudinally varying thicknesses [Greenspan, 1968]. Here we treat the Ekman layer as a simple linear boundary layer whose mean thickness is given by (3), and do not consider the dependence of this thickness on latitude. Furthermore, rotating convection in spherical shells is capable of driving strong zonal flows [e.g., Christensen, 2002; Heimpel et al., 2005]. Such flows are not typically observed in plane layer convection. Strong shearing in these zonal flows can influence heat transfer from inner to outer boundary [Aurnou et al., 2008]. Despite these possible complications, a meta-analysis by Aurnou [2007] shows that heat transfer scaling behavior may not strongly differ between the plane layer and spherical shell geometries.

[13] The presence of magnetic fields may also call into question the application of the plane layer, hydrodynamic results to dynamo models. For example, in studies with externally imposed magnetic fields, the Lorentz force acts to constrain fluid motions and inhibits heat transfer [Cioni et al., 2000; Aurnou and Olson, 2001]. However, in fully self-consistent dynamo models, heat transfer
scaling behavior is not found to differ significantly from that of comparable models without magnetic fields [Christensen, 2002; Christensen and Aubert, 2006; Aurnou, 2007].

[14] Another complication in applying the results of King et al. [2009] to dynamo models is introduced by the fact that the Ekman layer may no longer be an adequate representation of the viscous boundary layer. Instead, flows in the presence of magnetic fields and background rotation produce an Ekman-Hartmann layer [e.g., Desjardins et al., 2001]. This boundary layer balances the viscous force at the bounding surfaces with both Coriolis and Lorentz forces, instead of with only the Coriolis force alone as in the case of the Ekman layer. The Ekman-Hartmann layer scales as [Gilman and Benton, 1968; Debnath, 1973]

\[
\delta_{EH} \sim \left[ (\Lambda^2 + 1)^{1/2} - \Lambda \right]^{1/2} E^{1/2},
\]

where \( \Lambda \) is the Elsasser number. The Elsasser number characterizes the ratio of the strength of the Lorentz force to that of the Coriolis force, and is defined in Table 1. Figure 2 shows the ratio of the theoretical Ekman-Hartmann layer and Ekman layer thicknesses versus \( \Lambda \) for a uniform imposed transverse magnetic field in a plane fluid layer. The dynamo models presented here have a mean Elsasser number of 7.4, with extrema at 0.01 \( \leq \Lambda \leq 150 \). Assuming the boundary layer dynamics do not differ between flows with imposed magnetic fields and those with self-generated dynamo fields, Figure 2 illustrates that the thickness of the Ekman-Hartmann layer scales differently than the Ekman layer for the dynamo models with higher \( \Lambda \) values.

Therefore, the development of the transitional Rayleigh number from the nonmagnetic [King et al., 2009] study may not be appropriate for dynamo models.

[15] We also assume that the Rayleigh number at which convection onsets follows the theoretical scaling for rapidly rotating, nonmagnetic convection in a sphere, \( Ra_c \propto E^{-4/3} \) [Roberts, 1968; Busse, 1970; Jones et al., 2000; Dormy et al., 2004]. In the presence of strong imposed magnetic fields (\( \Lambda \gtrsim O(1) \)), the critical Rayleigh number has been shown to scale differently, as \( Ra_c \propto E^{-1} \) [Zhang, 1995; Fearn, 1998]. Studies of convection dynamics in the presence of imposed magnetic fields therefore suggest a significant difference from that found in hydrodynamic studies. For example, a critical Rayleigh number difference of \( E^{1/3} \) between magnetoconvection and nonmagnetic convection studies predicts an order of magnitude difference in \( Ra_c \) across our range of Ekman numbers. Assuming the magnetoconvection results apply to dynamo models, this may again invalidate the scalings given by King et al. [2009].

[16] We hypothesize that the boundary layer control of heat transfer regimes demonstrated in the nonmagnetic, plane layer convection study also applies to planetary dynamo models. This hypothesis makes the following two predictions. First, heat transfer occurs in two separate regimes, rapidly rotating and weakly rotating, with distinct empirical heat transfer scalings in each regime, \( Nu = (Ra/Ra_c)^{8/5} \) and \( Nu \sim Ra_c^{2/3} \), respectively. Second, the transition between regimes is determined by relative boundary layer thicknesses and is described by the transitional Rayleigh number, \( Ra_t = E^{-7/4} \). In this work, we test these predictions using a broad array of numerical dynamo models.

3. Numerical Method

[17] We consider three-dimensional convection of a Boussinesq, electrically conducting fluid in a spherical shell rotating about the axial \( z \) direction with fixed angular velocity \( \Omega \). The shell has isothermal boundaries with an imposed temperature contrast of \( \Delta T \) between inner and outer boundaries. Absolute temperature values are therefore arbitrary, as the temperature on the outer boundary is set to \( T(r = R_o) = 0 \) for convenience. Gravity is assumed to vary linearly with radius and has a value of \( g_s \) on the outer boundary. Figure 3 shows a schematic depiction of the dynamo model’s geometry.
Equations (1)–(4) are solved simultaneously for \( u, p, B, \) and \( T \) using the numerical model MagIC version 3.38 [Wicht, 2002]. This code makes use of the pseudospectral method first developed by Glatzmaier [1984] with subsequent modifications by Christensen et al. [1999] and Wicht [2002]. The velocity and magnetic induction vectors are decomposed into poloidal and toroidal scalar potentials, which are expanded in Chebyshev polynomials in the radial direction and in spherical harmonic functions on spherical surfaces. MagIC utilizes mixed implicit and explicit time stepping. The Coriolis and nonlinear terms are treated explicitly using a second-order Adams-Bashforth scheme and the diffusion, pressure, and linear terms are treated implicitly using a Crank-Nicolson time step. The implicit time step can vary over time and is controlled by a modified MHD Courant criterion which accounts for viscous and ohmic damping of short-wavelength Alfvén-type oscillations [Christensen et al., 1999]. This damping helps to stabilize the system and allows the use of a slightly larger numerical time step compared to the unmodified MHD Courant criterion. Hyperdiffusion is used in our most extreme cases to increase numerical stability by damping the small-scale components of the flow and magnetic fields. In these cases, the viscous, thermal, and magnetic diffusivities are multiplied by a factor \( d \) of the form

\[
d(l) = 1 + A \left( \frac{l + 1 - l_{HD}}{l_{\text{max}} + 1 - l_{HD}} \right)^{\beta}
\]

in the spherical harmonic domain. Here \( A \) is the hyperdiffusion amplitude, \( l \) is the spherical harmonic degree, \( l_{HD} \) is the degree above which hyperdiffusion starts to act, \( l_{\text{max}} \) is the maximum harmonic degree, and \( \beta \) is the hyperdiffusion exponent.

We analyze a data set consisting of twenty-one new simulations, hereon called the dynamo subset models; and one hundred seventy nine additional dynamos referred to as the dynamo factory models, most of which were previously reported by Christensen and Aubert [2006], Olson and Christensen [2006], and Christensen et al. [2009]. To our knowledge, this is the broadest array of planetary dynamo models currently available. The range of parameters explored by each is given in Table 1, and Table 2 details each dynamo model geometry. The convecting fluid is contained in a spherical shell between a hot inner boundary and cold outer boundary, \( T_i > T_o \). Gravity points radially inward and increases linearly with spherical radius. The rotation axis is vertical with uniform nondimensional rotation rate \( E^{-1} \).

Figure 3. A schematic illustration of the dynamo model geometry. The convecting fluid is contained in a spherical shell between a hot inner boundary and cold outer boundary, \( T_i > T_o \). Gravity points radially inward and increases linearly with spherical radius. The rotation axis is vertical with uniform nondimensional rotation rate \( E^{-1} \).
subset case, for which we have more comprehensive diagnostic capabilities. All simulations have an Earth core-like geometry (χ ≡ R/Ro = 0.4 for dynamo subset models, and χ = 0.35 for dynamo factory models) and rigid, no-slip, isothermal boundary conditions. The dynamo subset cases are full-sphere calculations with no presumed symmetries.

[21] The inner spheres of the models are either electrically insulating or having the same electrical conductivity as the outer fluid shell. The ratio of conductivities between the two regions in the dynamo subset is unity with one exception to test the importance of inner core conductivity. Cases C and IIC in Table 2 have identical input parameters but with conducting and insulating inner cores, respectively. The Nusselt numbers differ only by about 0.2%, so we can conclude that the inner core conductivity minimally affects the heat transfer behavior.

[22] The four most strongly driven subset cases use hyperdiffusion. For example, case H1 has A = 10, l_{HD} = 25, β = 2, l_{max} = 192, such that hyperdiffusion acts on spherical harmonic degrees above 25, increasing quadratically with degree. We test its influence by comparing two cases, with and without hyperdiffusion (cases H0 and H1, respectively, in Table 2). The efficiency of convective heat transfer (Nu) agrees within 0.2% between these two cases.

4. Analysis of Heat Transfer in Dynamo Models

4.1. Testing the Boundary Layer Control Hypothesis

[23] Figure 4 shows heat transfer data, Nu versus Ra, for all dynamo cases. The basic behavior of the heat transfer data is similar to that observed in plane layer studies [e.g., Rossby, 1969; King et al., 2009; Schmitz and Tilgner, 2009]. Convective heat transfer is strongly affected by the background rotation, whose strength is characterized by E⁻¹. The stabilizing influence of the Coriolis force requires models with higher rotation rates to be driven harder in order to convect [Chandrasekhar, 1953]. Thus, the onset of convection is delayed by the influence of rotation as E decreases. As in Figure 1, the heat transfer data follows two basic scaling behaviors. For sufficiently low E and Ra, the data appear to follow the rapidly rotating Nu ∝ Ra²/³ scaling (dashed line). In contrast, cases with higher E and Ra appear to transition to a weakly rotating Nu ∝ Ra⁻¹ scaling (solid line). Thus,
within particular parameter ranges, the data are described by either the rapidly rotating or nonrotating scaling exponents found in the plane layer study (Figure 1).

We aim to collapse the data in Figure 4 through a combination of control parameters. Figure 5 shows $Nu$ normalized by the nonrotating scaling ($Ra^{2/7}$) versus $Ra$ normalized by its predicted transitional value, $Ra_t = E^{-7/4}$, for all dynamo cases. The boundary layer transition scaling effectively collapses the broad array of dynamo heat transfer data. The scaling regimes hinted at by Figure 4 are more clearly observed in this parameter space defined by the boundary layer controlled transitional Rayleigh number.

Well into the rapidly rotating regime ($Ra \lesssim 0.1Ra_t$), heat transfer behavior are reasonably well described by the $Nu = (Ra/Ra_c)^{6/5}$ scaling (dashed line, (6)). In the weakly rotating regime ($Ra \gtrsim Ra_t$), heat transfer transitions to a $Nu \sim Ra^{2/7}$ scaling (solid horizontal line). A transitional region appears between the two scaling regimes, roughly in the range $0.1 \approx Ra/Ra_t \lesssim 2$.

Despite the added complexities of the spherical geometry and magnetohydrodynamic effects, the heat transfer regimes identified in plane layer, nonmagnetic convection (Figure 1) are also evident in these planetary dynamo models (Figures 4 and 5). The collapse of our data using $Ra_t$ shows no strong dependence on the Prandtl numbers, $Pr$ and $Pm$. Furthermore, our description of the heat transfer behavior in these dynamo models lacks any explicit dependence on the magnetic field strength; $Ra$, and $Ra_t$ depend on the rotation rate, but are independent of magnetic field intensity. Notably, this independence is at odds with studies of rotating convection of in the presence of imposed magnetic fields [e.g., Chandrasekhar, 1954; Nakagawa, 1959; Eltayeb and Roberts, 1970; Zhang, 1995; Aurnou and Olson, 2001; Fearn, 1998].

In order to compare the heat transfer transition to the boundary layers themselves, Figure 6 shows the velocity and thermal boundary layer thicknesses from the dynamo subset models plotted versus the transition parameter, $Ra/Ra_t$. The velocity boundary layer thickness is defined as the radial location of the first local maximum in RMS velocity above (below) the model’s inner (outer) boundary [Belmonte et al., 1994; King et al., 2009]. Similarly, the thermal boundary layer is defined as the radial location of the first local maximum in temperature variance away from the model’s boundaries [Belmonte et al., 1994; King et al., 2009]. The influence of magnetic fields does not affect our evaluation of the velocity boundary layer thicknesses, which are calculated directly from the velocity field in each subset case. Figure 6 shows that the two boundary layers cross where $Ra \approx Ra_t$. Thus, the transition from rapidly rotating to weakly rotating heat transfer behavior in the dynamo subset models corresponds well with the interchanging of the two nested boundary layers.
4.2. Asymptotic Scalings

Boundary layer control of heat transfer behavior in planetary dynamo models allows us to predict the limits of applicability of the various heat transfer scaling laws. In the weakly rotating regime, \( Ra > Ra_t \), the data conform to a

\[
Nu = 0.075 Ra^{2/7}
\]  

(13)
scaling, a result that is important for the convective regions of slowly rotating stars and planets [e.g., Palacios et al., 2006; Aurnou et al., 2007]. This result also implies that nonrotating, turbulent convective heat transfer may follow a $\alpha \approx 2/7$ scaling in the spherical geometry, like that found in plane layer convection [e.g., Castaing et al., 1989; Glazier et al., 1999]. Nonrotating, plane layer convection has the advantage of being carried out both numerically and in the laboratory. Laboratory convection studies are historically capable of achieving higher degrees of supercriticality than numerical models [e.g., Niemela et al., 2000]. However, high Rayleigh number, nonrotating convection has not been well explored in spherical geometries because of the difficulty in producing a spherically radial gravity field in the laboratory [cf. Tilgner, 1996].

Empirically, they find $\beta \approx 6/11$ [Christensen, 2002; Christensen and Aubert, 2006; Aurnou, 2007]. The range of parameter space in which the rotational time scale should be dominant is not quantified, however, and so the applicability of the diffusivity-free scaling is not well bounded.

[31] Figure 7 compares the goodness of fit of the diffusivity free scaling law with that of the rapidly rotating scaling (6) discussed above. The rapidly rotating scaling law (6) and the diffusivity free scaling law (16) can be rewritten in terms of base parameters as

$$\frac{Nu}{Ra^*} = \frac{RaNuE^3Pr^{-2}}{Nu^*} = \frac{Nu}{Ra^*}$$

respectively. Figure 7 shows heat transfer data from the dynamo models normalized by the two scalings given in (17) and (18). Rather than spreading the data across many orders of magnitude, as in $Nu^*-Ra^*$ plots, here we collapse the data to illustrate the limits of the scaling laws’ applicability. The rapidly rotating scaling (17) appears to collapse heat transfer data more effectively than the diffusivity free scaling (18). However, the lack of any clear convergence to a single scaling behavior in Figure 7a (for $Ra < Ra_c$) illustrates that no unambiguous asymptotic behavior has been reached in the rapidly rotating regime.

[32] This ambiguity in asymptotic behavior arises from the limited range of Rayleigh numbers currently accessible within the rapidly rotating regime ($Ra_c < Ra < Ra_s$). A significantly broader range of supercritical ($Ra > Ra_c$), subtransitional ($Ra < Ra_s$), Rayleigh numbers is needed to test the validity of this scaling. The breadth of Rayleigh number accessibility within the rapidly rotating regime scales with the Ekman number as

$$\frac{Ra_s}{Ra_c} \approx \frac{E^{-5/12}}{3.5}$$

For example, at an Ekman number of $10^{-5}$, the range of $Ra$ within the rapidly rotating regime is only $Ra/Ra_c \approx 35$. Thus, in order to produce a
broader range of Rayleigh numbers within this regime, lower Ekman number simulations must be carried out. Until a clear asymptotic regime of highly supercritical \((Ra/Ra_c, C^2)\) highly supercritical, \((Ra/Ra_t < 1)\) rapidly rotating convection is reached, the veracity of any such scaling law is questionable. The diffusivity-free heat transfer scaling (18) is one member of a suite of empirical, diffusivity-free scaling laws. Within the range of parameter space currently accessible, this rotationally dependent diffusivity-free approach has produced scalings for several other model outputs such as typical velocities, length scales, and magnetic field strengths [Christensen, 2002; Aubert, 2005; Olson and Christensen, 2006; Christensen et al., 2009]. These scaling laws implicitly take the rotation period as their dominant time scale, replacing the typical diffusion time scales present in the base parameters (Table 1). We have demonstrated here that heat transfer behavior is no longer controlled by rotation rate when \(Ra > Ra_t\) (Figures 5 and 7). Therefore, we postulate that \(Ra_t\) provides an upper bound for the application of the other diffusivity-free scaling laws of Aubert [2005], Christensen and Aubert [2006], Olson and Christensen [2006], and Christensen et al. [2009].

5. Mean Temperature Profiles and Thermal Mixing

The rapidly rotating and weakly rotating heat transfer regimes exhibit different global thermal signatures. Figures 8 and 9 illustrate the transition in the mean temperature structures between the regimes. The rapidly rotating regime maintains large-scale mean internal temperature gradients while the weakly rotating regime undergoes vigorous mixing that produces an adiabatic interior fluid. (Mixing is a scale-dependent concept; here we consider mixing on the largest length scales of the system.) Since we consider Boussinesq convection, an adiabatic profile corresponds to a statistically isothermal interior fluid. This isothermalization of bulk fluid by convection has been observed in nonrotating [Belmonte et al., 1994; Camussi and Verzicco, 1998] and weakly rotating convection [Aurnou et al., 2007; King et al., 2009], and is typically associated with chaotic, three dimensional turbulence. In rapidly rotating convection, however, the persistence of coherent thermal structures limits mixing, thereby permitting the establishment of significant nonzero mean temperature gradients within the bulk fluid [e.g., Julien et al., 1996; Gillet and Jones, 2006; Sprague et al., 2006; King et al., 2009]. Figure 8 also shows that mean, meridional temperature structures exist in the rapidly rotating regime. This implies that the influence of rotation may promote large-scale baroclinicity in convection [cf. Kaspi et al., 2009; Jones and Kuzanyan, 2009].

We quantify the internal temperature gradients through calculations of the mean temperature lapse rate (i.e., the radial gradient of the temporally averaged temperature field, \(dT/dr\)) at midshell for each of the dynamo subset models. A perfectly well mixed Boussinesq fluid will have a zero mean
Figure 8. Meridional slices of time-averaged temperatures \((T - T_o)/\Delta T\) from the dynamo subset. (a) Conductive end-member case for convective mixing, from (20). Cases (b) A, (c) B, (d) C, and (e) D in Table 2. Figures 8b and 8c show cases in the rapidly rotating regime \((Ra < Ra_t)\); Figures 8d and 8e show cases in the weakly rotating regime \((Ra > Ra_t)\). (f) Isothermal end-member case for convective mixing, from (28).

Figure 9. Mean radial temperature profiles from the dynamo subset cases shown in Figure 8. Temperature is normalized as \((T - T_o)/\Delta T\), and radial position is normalized by the outer boundary radius. The dotted line indicates the conductive temperature profile from (20), and the dashed line represents the idealized isothermal temperature profile, \(T = T_o + 0.14\Delta T\), from (28) with \(\chi = 0.4\). Note that the internal temperature gradients approach zero as \(Ra/Ra_t\) increases.
Figure 10. (a) Nondimensional temperature lapse rate, $|\frac{dT}{dr}|$, at midshell ($r = (R_i + R_o)/2$) versus the boundary layer transition parameter, $Ra/R_o$, for the dynamo subset. The dotted line represents the conductive lapse rate at midshell (using (20) and $\chi = 0.4$): $\frac{dT}{dr} = -\Delta T \frac{R_o R_i}{R_o - R_i} r^{-2} = -0.82 \frac{\Delta T}{R_o - R_i}$. (b) Mean internal temperature, $\langle (T) - T_o \rangle / \Delta T$, versus the boundary layer transition parameter, $Ra/R_o$, from the dynamo subset models. The dotted line indicates the mean temperature of a conductive shell, $\langle T \rangle = T_o + 0.23 \Delta T$, from (21) with $\chi = 0.4$. The dashed line represents a mean temperature estimate for well-mixed convection, $\langle T \rangle = T_o + 0.14 \Delta T$, from (28) with $\chi = 0.4$.

temperature gradient in the interior fluid, $dT/dr = 0$, such that the bulk fluid is isothermal. As the influence of rotation grows (for decreasing $Ra/R_o$), some rigidity is imparted on the flow via the Coriolis force, and increasingly strong thermal gradients can abide, as shown by King et al. [2009].

[36] Figure 10a shows measurements of midshell lapse rates from the dynamo subset. We observe a transformation from a nearly conductive thermal signature in the rotationally controlled regime ($Ra < Ra_o$) to a nearly isothermal interior fluid in the weakly rotating regime ($Ra > Ra_o$). The transition to the nonrotating heat transfer scaling, $Nu \sim Ra^{2/7}$, is therefore linked to the elimination of mean interior temperature gradients by efficient convective mixing. In the rapidly rotating heat transfer regime, however, the persistence of mean temperature gradients indicates that strong thermal mixing is inhibited by the influence of rotation.

[37] The lapse rate behavior in Figure 10a illustrates the relationship between heat transfer regimes and thermal mixing in a limited subset of our dynamo model data. In order to investigate mixing more systematically, we would like to quantify the strength of mean thermal gradients in the broader array of dynamo factory models. However, we do not have lapse rate data for these models. Instead, we use volume-averaged temperature calculations as a proximate measure of internal temperature gradients.

[38] Figure 10b shows mean internal temperatures from the dynamo subset models. The mean internal temperature is the average temperature over the volume of the shell, $\langle (T) - T_o \rangle / \Delta T$, where $\langle \cdot \rangle$ indicates volume averaging. A comparison of Figures 10a and 10b shows that the lapse rates and mean internal temperatures display qualitatively similar behavior. In the following paragraphs, we relate mean temperature to the efficiency of thermal mixing by comparing model data against the two idealized end-member cases: a conductive and an isothermal temperature profile.

[39] First, in an incompressible spherical shell with isothermal boundaries, the conductive case will have a temperature profile of the form [Kono and Roberts, 2001]:

$$T(r) = T_o + \left( \frac{R_o R_i}{R_o - R_i} \left( \frac{1}{r} - \frac{1}{R_i} \right) \right) \Delta T,$$

where $R_o$ and $R_i$ are the radii of the outer and inner boundaries, respectively, and $T_o$ and $T_o + \Delta T$ are the imposed outer and inner boundary temperatures, respectively. This expression can also be written in terms of the shell’s radius ratio, $\chi = R_o/R_i$ as

$$T(r) = T_o + \left( \frac{\chi}{\chi - 1} \left[ 1 - (r/R_i)^{-1} \right] \right) \Delta T.$$

We can integrate (20) over the volume of the shell to solve for the mean temperature for a conductive fluid:

$$\langle T_{\text{conductive}} \rangle = T_o + \left( \frac{R_o + 2R_i}{R_o^2 + R_o R_i + R_i^2} \right) \frac{\Delta T}{2}.$$

This expression can also be written in terms of the shell’s radius ratio as

$$\langle T_{\text{conductive}} \rangle = T_o + \left( \frac{\chi (1 + 2\chi)}{1 + \chi + \chi^2} \right) \frac{\Delta T}{2}.$$
The temperature drop across the inner and outer thermal boundary layers is equal thickness, which we qualitatively observe to hold true in dynamo subset temperature profiles. Furthermore, the collapse of mean temperature data indicates that boundary layer processes are directly related to the interior temperature field.

Convective regions in planets and stars are typically assumed to be well mixed on account of their strongly turbulent nature. The Reynolds number, \( Re \), compares inertial and viscous forces on global length scales, and is thought of as a parameterization of the degree of turbulence (see Table 1 for definition). High \( Re \) fluids in the absence of rotation are dispersive media wherein the viscous dissipation is diminished in planetary dynamo models.
gradients of passive scalars are quickly smoothed [McWilliams, 2006]. We may expect, then, that the thermal mixing observed in the dynamo models may be strongly influenced by $Re$.

Figure 12b shows the mean temperature measurements as in Figure 12a, but now plotted against the Reynolds number. We observe no dependence of thermal mixing on $Re$. Instead, a comparison of Figures 12a and 12b shows that $Ra/Ra_c$ and not the Reynolds number, characterizes thermal mixing in dynamo models.

6. Application to Earth’s Core

6.1. Convective Regime of the Core

Assuming the empirical scaling laws derived here apply to Earth’s core, we can estimate the Rayleigh number and heat transfer regime of the core. Estimates of the total superadiabatic heat flux escaping the core mantle boundary (CMB) typically fall in the range 1–10 TW [e.g., Buffett, 2003]. We can use these heat flux estimates, along with estimates of the thermophysical properties of core fluid, to determine the flux Rayleigh number in the core. The flux Rayleigh number is defined as

$$R_f = Ra Nu = \frac{\alpha_T g Q L^4}{A \rho c_P \nu \kappa^2},$$

where $Q$ is the total superadiabatic heat flux, $A$ is the surface area of the CMB, and $c_P$ is the specific heat of the core fluid. The flux Rayleigh number has the advantage of depending on $Q$, rather than the superadiabatic temperature drop across the core, $\Delta T$, which is less well constrained from geophysical observations.

Estimates of core properties are listed in Table 3. Given these properties, the flux Rayleigh number is estimated to lie in the range

$$R_f \approx (2.4 - 24) \times 10^{28} \quad (32)$$

Table 3. Estimates of the Physical Properties of Earth’s Core

<table>
<thead>
<tr>
<th>Property</th>
<th>Core Estimate</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_T$</td>
<td>$10^{-3} \text{ K}^{-1}$</td>
<td>Gubbins [2001]</td>
</tr>
<tr>
<td>$g$</td>
<td>10 m/s$^2$</td>
<td>Dziewonski and Anderson [1981]</td>
</tr>
<tr>
<td>$Q$</td>
<td>$(1-10) \times 10^{12} \text{ W}$</td>
<td>Buffett [2003]</td>
</tr>
<tr>
<td>$L$</td>
<td>2260 km</td>
<td>Dziewonski and Anderson [1981]</td>
</tr>
<tr>
<td>$A$</td>
<td>$1.5 \times 10^{14} \text{ m}^2$</td>
<td>Dziewonski and Anderson [1981]</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$10^4 \text{ kg/m}^3$</td>
<td>Dziewonski and Anderson [1981]</td>
</tr>
<tr>
<td>$c_P$</td>
<td>700 J/kgK</td>
<td>Gubbins [2001]</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$10^{-6} \text{ m}^2/\text{s}$</td>
<td>de Wijs et al. [1998]</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>$10^{-5} \text{ m}^2/\text{s}$</td>
<td>Gubbins [2001]</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>$7.3 \times 10^{-5} \text{ rad/s}$</td>
<td>McCarthy and Babcock [1986]</td>
</tr>
</tbody>
</table>

*Properties are as follows: $\alpha_T$ is the fluid’s coefficient of thermal expansivity, $g$ is gravitational acceleration, $Q$ is total superadiabatic heat flux, $L$ is the distance between inner core boundary (ICB) and core-mantle boundary (CMB), $A$ is the surface area of the CMB, $\rho$ is the density of the fluid, $c_P$ is the fluid’s specific heat, $\nu$ is the kinematic viscosity of the fluid, $\kappa$ is the fluid’s thermal diffusivity, and $\Omega$ is the angular rotation rate.
in Earth’s core. We can also use these core property estimates to determine an Ekman number for the core:

\[ E \approx 1.3 \times 10^{-15}. \]

[37] The heat transfer regime of the core is determined by comparing the estimated core \( R_f \) value with a transitional flux Rayleigh number, which can be written as

\[ R_f \approx Ra_t \nu_a \approx E^{-9/4}, \]

and has an approximate value of \( R_f \approx 2 \times 10^{32} \) in the core. The appropriate heat transfer scaling law for \( R_f < R_f^* \) is \( \nu \approx (Ra/Ra_t)^{6/5} \), which, together with our estimate of \( R_f^* \), allows us to solve for the range of Rayleigh and Nusselt numbers for core convection:

\[ Ra \approx (1 - 3) \times 10^{24} \quad ; \quad \nu \approx (2 - 8) \times 10^4. \]

Note that these values correspond to the diabatic component of core thermodynamics. For example, heat conducted down the core adiabat is estimated to be on the same order as that transported by convection, giving a different formulation of the Nusselt number, a total Nusselt number, of approximately 2 [e.g., Buffett, 2003].

[38] Using the estimate for \( E \) given above, the transitional Rayleigh number in the core is \( Ra_t = E^{-7/4} \approx 10^{26} \). We then estimate that \( Ra/Ra_t \) in the core is roughly in the range:

\[ Ra/Ra_t \approx (1 - 3) \times 10^{-2}. \]

Thus, our results suggest that core convection occurs in the rapidly rotating regime, but only just barely.

[39] As mentioned in section 2.1, previous work has sought to separate these two regimes based on measurements of the vigor of convective flow in various models [Olson and Christensen, 2006; Schmitz and Tilgner, 2009]. Olson and Christensen [2006] extrapolate scalings of local Rossby numbers in planetary dynamo models to predict that Earth’s core falls just before the transition from rotationally dominated magnetic field morphology to that nearly unaffected by rotation. Schmitz and Tilgner [2009] extrapolate calculations of a modified Peclet number from rotating convection simulations to Earth’s core using estimates of convective speeds from geomagnetic secular variation inversions. The authors predict that core convection lies within a transitional region between rapidly rotating and weakly rotating regimes. Thus, though by different means, the present work and these previous studies all place core convection near, but below, the nonrotating style convective regime.

[50] We speculate that Earth’s proximity to the transition is important for dynamics such as magnetic field morphology and polarity reversals. In particular, the rough alignment of Earth’s dominant dipole and rotation axis suggests that core flow is strongly influenced by rotation. However, the indirect observation of geomagnetic polarity reversals from seafloor striping hints at strong nonlinearities typical of more chaotic fluid dynamics. Our estimate that core convection resides in the rapidly rotating regime, but near the transition to weakly rotating behavior, provides a possible explanation of this dichotomy. Studies of polarity reversals in dynamo models find that the generation of predominantly dipolar, reversing magnetic fields typically occurs in a relatively narrow range of parameters between rotationally dominated and buoyancy dominated dynamo regimes [e.g., Wicht et al., 2009; Driscoll and Olson, 2009].

[51] Convection in Earth’s core is likely driven by some combination of thermal and compositional buoyancy [e.g., Braginsky and Roberts, 1995]. Compositional convection may arise as lighter, alloy rich liquid metal is formed near the bottom of the core from the solidification of the inner core. Similarly, chemical reactions at the core mantle boundary may generate compositionally heavy fluid. The chemical diffusivities of these fluid mixtures may be lower than that expected of thermal diffusivity in the core by about three orders of magnitude [e.g., Gubbins, 2001]. This would result in larger estimates of the effective core Prandtl number and Rayleigh number (and therefore \( Ra/Ra_t \)). The simulations presented here consider only thermal convection. It is unclear how the inclusion of compositional convection dynamics would alter our results [cf. Kutzner and Christensen, 2000], especially in regard to the arguments derived from boundary layer physics.

### 6.2. Convective Mixing in the Core

[52] We can also apply the analysis of thermal mixing in dynamo models (section 5) to Earth’s core. It is typically assumed that the strong turbulence in the core, where \( Re \sim 10^8 \) [Bloxham and Jackson, 1991], effectively mixes the bulk core fluid [e.g., Stacey, 1969; Stevenson, 1987; Braginsky and Roberts, 1995; Jones, 2009]. Here we show that thermal mixing is not strongly dependent on the
degree of turbulence as characterized by the Reynolds number. Instead, the efficiency of mixing by convection is linked to the relative importance of rotation. We estimate that Earth’s core lies in the rapidly rotating regime, since \( Ra < Ra_c \). Through measurements of the mean temperature of dynamo models, we have shown that significant superadiabatic density gradients persist in the rapidly rotating regime, independent of the vigor of flow, \( Re \). This implies that large-scale superadiabatic density anomalies may survive in the core, which would not be permitted by considerations of turbulent mixing without rotation. That is, rapidly rotating convection systems, like Earth’s core, may not be strictly adiabatic.

Interestingly, deviations from expected density profiles have been inferred seismically in the core [e.g., Romanowicz and Breger, 2000; Soldati et al., 2003]. Notable features include thick (\(~300–400\) km) regions of anomalous seismic wave velocities at the top [Souriau and Poupinet, 1991] and bottom [Kaneshima et al., 1994] of the core. Furthermore, recent seismic studies have shown that observationally significant temporal fluctuations in density occur in Earth’s outer core [Dai and Song, 2008]. It has been argued that such observations may be due to perturbations of geopotential surfaces in the core by density heterogeneity in the mantle [Wahr and de Vries, 1989]. However, it does not appear that this argument can explain the decadal time scale variations observed by Dai and Song [2008].

We can estimate the total superadiabatic density drop across the core using our estimate of the Rayleigh number (35) and the thermophysical properties listed in Table 3. The superadiabatic temperature drop across the outer core is then \( \Delta T \approx 10^{-2} \) K, giving a total superadiabatic density change of \( \Delta \rho/\rho_o \approx \alpha_f \Delta T \approx 10^{-7} \). This estimate represents a rough upper bound for mean density anomalies that may persist in the bulk of the outer core, and cannot account for the amplitude of density variations inferred from several of the seismic studies, which may be as large as \( \Delta \rho/\rho_o \approx 10^{-3} \) [Soldati et al., 2003; Dai and Song, 2008].

### 7. Summary

We hypothesize that heat transfer results found in the nonmagnetic, plane layer convection study of King et al. [2009] also applies to planetary dynamo models. This hypothesis predicts that heat transfer occurs in two separate regimes, rapidly rotating and weakly rotating, with empirical heat transfer scalings in each regime, \( Nu \sim (Ra/Ra_c)^{6/5} \) and \( Nu \sim Ra^{2/7} \), respectively. Second, the transition between regimes is determined by relative thicknesses of the thermal and velocity boundary layers, and is described by the transitional Rayleigh number, \( Ra_t = E^{-7/4} \).

We test this hypothesis using a broad array of numerical planetary dynamo models, and our heat transfer scaling results are summarized in Table 4. Heat transfer behavior in the dynamo models does exist in two distinct regimes, which are adequately separated by the transitional Rayleigh number (Figure 5). In the rapidly rotating regime \( (Ra \approx Ra_c) \), heat transfer is described by the scaling \( Nu = (Ra/Ra_c)^{6/5} \), where \( Ra_c = 3.5E^{-4/3} \). In the weakly rotating regime \( (Ra \approx Ra_t) \), heat transfer follows a \( Nu = 0.075Ra^{2/7} \) scaling. The \( Nu \sim Ra^{2/7} \) scaling implies that heat transfer in weakly rotating spherical systems is similar to that in nonrotating planar convection. In the rapidly rotating regime, \( Nu \) is significantly less than that expected for nonrotating convection because the influence of rotation inhibits convective heat transfer. Interestingly, our collapse of the heat transfer behavior is not explicitly dependent on magnetic field strength of these dynamo models.

Based on the boundary layer thickness calculations shown in Figure 6, we argue that the transition between these regimes is governed by the relative thicknesses of the thermal and velocity boundary layers. Thus, despite numerous differences between the convection system examined by King et al. [2009] and the dynamo models analyzed in this work, the basic heat transfer behavior of the two studies is in agreement.

Models in each of the two heat transfer regimes produce different thermal profiles. The rapidly rotating regime is distinguished by signifi-
cant, large-scale thermal gradients. Therefore, the strong influence of rotation on core convection may imply that seismically observable density structures can exist. In contrast, models residing in the weakly rotating regime are well mixed, producing nearly isothermal interiors. Thermal mixing is shown to depend on the boundary layer controlled transition parameter, $Ra/Ra_0$ (Figure 12a), and is not observed to depend on the Reynolds number (Figure 12b). Our results appear to imply that rotational effects on convective mixing may provide a dynamical mechanism for the seismically inferred density heterogeneities in the outer core [e.g., Romanowicz and Breger, 2000; Soldati et al., 2003]. However, when extrapolated to core conditions, our heat transfer scalings predict a Rayleigh number of $Ra \approx 2 \times 10^{14}$ in the core, corresponding to a superadiabatic density variation of $\Delta \rho/\rho_0 \approx 10^{-7}$, which cannot explain the seismically inferred anomalies.

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